Sporadic randomness: The transition from the stationary to the nonstationary condition

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We address the study of sporadic randomness by means of the Manneville map. We point out that the Manneville map is the generator of fluctuations yielding the Lévy processes, and that these processes are currently regarded by some authors as statistical manifestations of a nonextensive form of thermodynamics. For this reason we study the sensitivity to initial conditions with the help of a nonextensive form of the Lyapunov coefficient. The purpose of this research is twofold. The former is to assess whether a finite diffusion coefficient might emerge from the nonextensive approach. This property, at first sight, seems to be plausible in the nonstationary case, where conventional Kolmogorov-Sinai analysis predicts a vanishing Lyapunov coefficient. The latter purpose is to confirm or reject conjectures about the nonextensive nature of Lévy processes. We find that the adoption of a nonextensive approach does not serve any predictive purpose: It does not even signal a transition from a stationary to a nonstationary regime. These conclusions are reached by means of both numerical and analytical calculations that shed light on why the Lévy processes do not imply any need to depart from the adoption of traditional complexity measures.

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I. INTRODUCTION

In recent few years there has been an increasing interest in the possibility of establishing a connection between thermodynamics and dynamics through a generalization of the concept of sensitivity to initial conditions from exponential to power law [1–3]. The main idea behind these papers is that, according to the earlier work of Tsallis [4], the conventional Boltzmann-Gibbs-Shannon prescription

$$S=-\sum_i p_i \ln p_i \tag{1}$$

has to be replaced by

$$S_q=\frac{1-\sum_i p_i^q}{q-1}. \tag{2}$$

The meaning of Eq. (1) is well known. The classical phase space is divided into a discrete number of cells, and each cell is assigned a probability p_i. The quantity S, expressed in terms of the p_i's, according to Eq. (1), is the entropy of the dynamic system under study. The quantity S_q of Eq. (2) is a nonextensive generalization of the conventional entropy of Eq. (1). In fact, it is straigthforward to prove that S_q of Eq. (2) becomes identical to S of Eq. (1) when the parameter q, referred to as the *entropic index*, is assigned the value $q=1$. It is believed that the *power law* sensitivity to initial conditions at the edge of chaos provides a natural link between the entropic index q and the nonexponential sensitivity to initial conditions. To illustrate the heuristic arguments of Refs. [1–3], we can adopt the results of a more recent paper [5]. In this paper we illustrate a generalization of the Pesin theorem [6] on the basis of the nonextensive entropy of Eq. (2). As is well known [7,8], the Pesin theorem does not refer to the physical entropy, but to the entropy of a single trajectory. The phase space is divided into cells; each cell is assigned a label, and a single trajectory running on this phase space results in a sequence of symbols (the cell labels). We then consider a window of size N, move it along a symbolic sequence, and consider all possible combinations of N symbols corresponding to each window position. This makes it possible to evaluate the corresponding probability, and, by means of the adoption of the conventional entropy of Eq. (1), the entropy $S(N)$ corresponding to a window of size N. The Komogorov-Sinai (KS) entropy h_{KS} is defined [7,8] by

$$h_{KS}=\lim_{N\to\infty} S(N)/N. \tag{3}$$

The Pesin theorem [6] establishes an attractive connection between the KS entropy and an important dynamic property called the Lyapunov coefficient. To illustrate the meaning of this theorem, let us consider, for simplicity, a one-dimensional map, defined in the interval [0,1] as

$$x_{n+1}=\Phi(x_n). \tag{4}$$

The Lyapunov coefficient reads

$$\lambda(x_0)=\lim_{N\to\infty}\frac{1}{N},\Lambda(N,x_0), \tag{5}$$

with

$$\Lambda(N,x_0)=\sum_{n=0}^{N-1}\ln|\Phi'(x_n)|=\ln\prod \Phi'(x_n). \tag{6}$$

The Pesin theorem states that if the invariant distribution $\rho(x)$ exists, the KS entropy reads

$$h_{KS}=\lambda=\frac{\int_0^1 \ln|\Phi'(x)|\rho(x)dx}{\int_0^1 \rho(x)dx}. \tag{7}$$

It is evident that this theorem is a natural consequence of adopting the form of Eq. (6) supplemented by the ergodic assumption.

In the case where the condition of strong chaos does not apply, the linear dependence of $S(N)$ on N, implied by Eq. (3), can be lost. Thus it seems to be convenient to generalize the KS approach by using the entropy of Eq. (2) rather than that of Eq. (1). In this case we would replace $S(N)$ with $S_q(N)$. Let us focus our attention on the case $N \gg 1$, namely, on windows of extremely large size, and let us denote this virtually continous time by the symbol t. In this case, according to the authors of Ref. [5], for $S_q(t)$ we obtain the following expression:

$$S_q(t)\equiv\frac{1-\delta^{q-1}\int dx\rho(x)^q\xi(t,x)^{1-q}}{q-1}. \tag{8}$$

The theory of Ref. [5] rests on the division of the phase space into cells, so as to make possible an experimental determination of probability density; the symbol δ denotes the cell size. The function $\xi(t,x(0))$, in fact, is defined, by the prescription

$$\xi(t,x(0))\equiv\lim_{\Delta x(0)\to 0}\frac{\Delta x(t)}{\Delta x(0)}, \tag{9}$$

where $\Delta x(t)$ denotes the distance at time t between two trajectories that at the initial time $t=0$ are located at a distance $\Delta x(0)$ from one another. The theoretical result of Eq. (8) makes it easy for us to explain why the nonextensive generalization of the KS entropy can make $S_q(t)$ increase linearly in time, even if the ordinary exponential sensitivity to initial condition is lost and is replaced by a power law sensitivity. To show this important property, let us consider the simplifying condition where $\xi(t,x)$, with a power law form, is not x dependent. In this case we have

$$\xi(t)=[1+(1-Q)\lambda_Q t]^{1/(1-Q)}. \tag{10}$$

Under the simplifying conditions adopted, the coefficent λ_Q, is independent of x. As shown hereby, the coefficent λ_Q plays the role of generalized KS entropy. To see this property, let us replace $\xi(t)$ in Eq. (8). Let us assign, to the entropic index q, the ‘‘magic’’ value $q=Q$, corresponding to the sensitivity to initial conditions of Eq. (10). In this condition $S_q(t)$ of Eq. (8) becomes linearly dependent on time. In accordance with Eq. (3), we must define the nonextensive form of the KS entropy as the limit for $t\to\infty$ of $S_q(t)/t$. The limiting value in this case is given by the coefficent λ_Q. This is the reason why the authors of Refs. [1–3] imaged a connection between thermodynamics and dynamics more extended than that based on an exponential sensitivity to initial conditions.

The main purpose of this paper is to assess whether or not this proposed generalization of the conventional approaches might afford some benefit in the case of intermittent processes. First of all, we note that this attractive result is obtained by interpreting the window size as ‘‘time,’’ so as to establish a connection with the function of time $\xi(t)$ of Eq. (10), that the authors of Refs. [1–3] considered to be the key ingredient to generalize the connection between dynamics and thermodynamics. To make our analysis of their conjectures unambiguous, we adopt a safer point of view, according to which N plays the same role as the volume of thermodynamic formalism. Thus we rest on an *ad hoc* generalization of Eq. (6). The rationale for this choice is as follows. The expected generalized form of sensitivity of Eq. (10) suggests a definition of

$$\Xi\ \exp_q(x)\equiv[1+(1-q)x]^{1/(1-q)}, \tag{11}$$

which implies a definition of the q logarithm:

$$\ln_q(x)\equiv\frac{x^{1-q}-1}{1-q}. \tag{12}$$

Thus the analog of Eq. (5) becomes:

$$\lambda_q(N,x_0)=\lim_{N\to\infty}\frac{1}{N}\Lambda_q(N,x_0), \tag{13}$$

with

$$\Lambda_q(N,x_0)\equiv\frac{1}{1-q}\left[\left(\prod_{n=0}^{N-1}|\Phi'(x_n)|\right)^{(1-q)}-1\right]. \tag{14}$$

This is the generalized form of the Lyapunov coefficient that will be studied in this paper. For computational convenience, we shall also make an average over a distribution of initial conditions.

The purpose of this paper is twofold. The first is to see if the Lyapunov coefficient of Eq. (14) can be finite in the region where the ordinary KS entropy vanishes. The second is to decide whether or not the Lévy processes can really be interpreted as manifestations of the Tsallis nonextensive thermodynamics. With the help of the Manneville map illustrated in Sec. II, we shall explain these two motivations more clearly.

II. MANNEVILLE MAP

The case of sporadic randomness under study in this paper is the Manneville map [9]. This maps reads

$$x_{n+1}=\Phi(x_n)=x_n+x_n^z(\mathrm{mod}\ 1)\quad (z\geqslant 1). \tag{15}$$

For the sake of the reader’s convenience, we show this map in Fig. 1. It is characterized by a laminar region given by the interval $[0,d]$, with d defined by the equation.

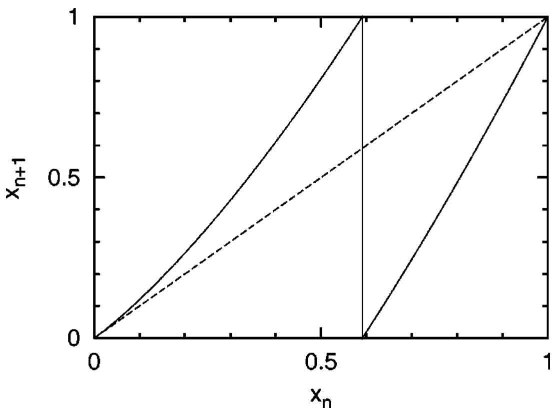

FIG. 1. The Manneville map. This is the form corresponding to $z=1.7$. The vertical full line divides the laminar region, on the left, from the chaotic region, on the right. The abscissa of the dividing line, d, is determined by Eq. (16) and, in the case of this figure, turns out to be $d\approx 0.59$.

$$d^z+d=1. \tag{16}$$

This region is illustrated in Fig. 1 by the portion on the left of the full vertical line. In this region the time evolution of a trajectory can be approximated by the continuous equation

$$\dot{x}=x^z. \tag{17}$$

The region on the right of the vertical full line is the chaotic portion of the map that, as we shall see in this paper, is the main source of the entropy increase. When the trajectory enters the chaotic region, after a few steps it is randomly injected back into the laminar region. As we shall see in Sec IV, an analytical treatment is possible if we make the assumption that the chaotic region is confined at the point $x=1$, so that the continuous approximation of Eq. (17) extends to the whole interval $[0,1]$. In this case the Pesin theorem leads us to

$$h_{KS}=\frac{\int_0^1 dx \frac{1}{x^{z-1}} \ln(1+zx^{z-1})}{\int_0^1 dx \frac{1}{x^{z-1}}}. \tag{18}$$

At $z=1$ this expression yields $\ln 2$, in accordance with the fact that the Manneville map becomes identical to the Bernoulli map. At $z=2$ the invariant distribution becomes equivalent to a δ of Dirac located at $x=0$, thereby yielding the vanishing value. It is therefore reasonable that a good approximation to Eq. (18) is given by

$$h_{KS}=z(2-z)\ln 2. \tag{19}$$

This is confirmed by Fig. 2, where we see that the approximated analytical expression of Eq. (19) fits the results of a numerical evaluation of Eq. (18) very well. We also see that both predictions are close to the KS entropy of the Manneville map evaluated by Gaspard and Wang [10], based on the use of Eq. (5). This supports the adoption of the theory of Sec. IV, which is based on approximations yielding Eq. (18).

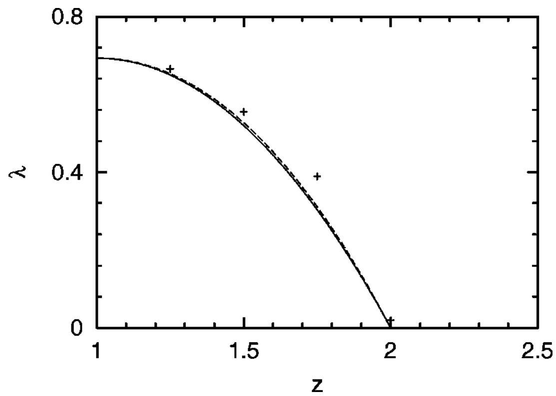

FIG. 2. The Lyapunov coefficent λ of Eq. (5) as a function of z. The dotted line denotes the numerical value of the integral of Eq. (18). The full line, coinciding in this scale with the dotted line, denotes the approximated expression of Eq. (19). The dot denotes the results of the numerical simulation of Ref. [10].

We are now in a position to explain the first of the two issues that we aim to settle in this paper. We see from Fig. 2 that the KS entropy vanishes in the region $z>2$. The former issue corresponds to the following question: Is it also possible to obtain a finite Lyapunov coefficient for $z>2$, if we adopt the non-extensive perspective advocated by Tsallis?

As far as the latter issue is concerned, for a proper illustration we must briefly review earlier work [11]. The Manneville map can be regarded as a dynamic generator of Lévy processes. As explained in Refs. [11] and [12], to derive a process of diffusion from the Manneville map we can proceed as follows. Let us consider the time at which the trajectory is injected in the laminar region. We record the time steps necessary for the trajectory to reach the border with the chaotic region. We build up a sequence of positive numbers W (or of the negative numbers $-W$, if we wish), by assigning to each time step the symbol W (or $-W$). The time spent after the exit into the chaotic region is very short, and can be neglected. After injection back into the laminar region we generate a sequence of $-W$ (of positive numbers W if earlier we made the choice of $-W$). The next time we shall come back to a sequence of W ($-W$), and so on. We can imagine the resulting sequence of W's and $-W$'s as the velocity fluctuations of a particle moving on a real axis y. This is a discrete realization of the diffusion process described by the equation of motion

$$\dot{y}=\eta(t), \tag{20}$$

where the fluctuating variable $\eta(t)$ has only two possible values, either W or $-W$. It was noted [11] that if we look at a trajectory moving in y space as a form of random walker, this random walker has a probability $\Pi(|y|)$ of making a jump of length $|y|$ which is given by

$$\Pi(|y|)=\psi(|y|/W)/W, \tag{21}$$

where $\psi(t)$ (with $t>0$) is the distribution of times of sojourn in the laminar region. For the Manneville map the function $\psi(t)$ turns out [11] to be

$$\psi(t)=d^{z-1}[1+d^{z-1}(z-1)t]^{z/(1-z)}. \tag{22}$$

This means that the region $1.5<z<2$ corresponds to a case where the first moment of the waiting time distribution is finite, whereas the second moment diverges. Note that for the system to make a jump of intensity $|y|$ it takes a time $t=|y|/W$. This is a form of Lévy walk that was proven to become equivalent to a Lévy flight [13]. According to the perspective established by Ref. [11], Eq. (21) explains why the advocates of nonextensive thermodynamics [14–16] claim that Lévy statistics imply nonextensive thermodynamics. In fact, the transition probability $\Pi(|y|)$ can be derived from a procedure of entropy maximization [11]. This transition probability is related to the waiting function distribution of Eq. (22) by the relation of Eq. (21). Thus, the waiting function of Eq. (22) also can be derived from a procedure of entropy maximization. The authors of Refs. [14–16] applied this method, setting constraints on the norm and on the second moment. The ''microcanonical'' treatment of Ref. [11] implies that we set a constraint on the first moment of $\psi(t)(t>0)$. The process of entropy maximization by means of the entropy [Eq. (2)], with the generic entropic index q, for $\psi(t)$ yields a form of the power law dependence $[A+Bt]^{1/(q-1)}$, where A and B are constants whose explicit expression was given in Ref. [11]. By comparing this expression to Eq. (22), we immediately obtain

$$q=1+(z-1)/z. \tag{23}$$

This is not a significant result by itself. It is essentially equivalent to an arbitrary labeling of the power index. The ambitious aim of the authors of Refs. [1–3] was to extend the connection between thermodynamics and dynamics from the ordinary case of exponential sensitivity to that of power law sensitivity. For this purpose to be satisfactorily realized, it is necessary to prove that the entropic index q is ''magic'' in the sense earlier pointed out. An attractive possibility was discovered in Ref. [11]. This had to do with the fact that the Manneville map results in the following analytical expression for the function $\xi(t)$:

$$\xi(t)=[1-(z-1)x^{z-1}t]^{-z/(z-1)}. \tag{24}$$

If we compare this analytical expression to Eq. (10), at first sight we are led to conclude that the magic value of the entropic index q is given by

$$Q=1+(z-1)/z, \tag{25}$$

in accordance with the entropic argument yielding Eq. (23). In fact, as pointed out in Sec. I, with this choice of the entropic index the entropy of Eq. (8) increases linearly in time, a property implying a steady rate of entropy increase similar to the standard KS condition. This would be a strong support of the conjecture made by the authors of Refs. [1–3], and would confirm that Lévy statistics imply indeed a nonextensive form of thermodynamics.

Unfortunately the authors of Ref. [11] could not reach this interesting conclusion, because a careful comparison of Eqs. (24) and (10) shows that the assumption of independence on initial condition is wrong, and that

$$\lambda_Q=zx^{z-1}, \tag{26}$$

thereby implying a dependence on x that makes questionable the connection between dynamics and thermodynamics. We shall see that this questionable connection is caused by the conviction that the direct origin of thermodynamic behavior is the sensitivity to initial condition. The form that the function $\xi(t)$ of Eq. (9) gains during motion within the laminar region does not have a direct effect on thermodynamics. The peculiar nature of the resulting thermodynamics is determined rather by the sporadic action of the chaotic region. This results in a sort of reduction of the rate of entropy increase of each trajectory, which becomes stronger and stronger, with z approaching the critical value $z=2$. At this critical value a transition to another regime takes place. This regime is not stationary, in the sense that the distribution density keeps moving towards a sort of Dirac δ located at $x=0$ with an infinite time scale, and any connection with Lévy statistics is lost.

III. NUMERICAL RESULTS

The remarks of Secs. I and II do not leave any room for the adoption of a nonextensive perspective as a form of equilibrium thermodynamics. In fact, if we adopt the equilibrium perspective, as illustrated by Fig. 2 and Eq. (19), we are forced to accept the ordinary perspective associated with the existencce of a finite Lyapunov coefficent for $z<2$. This range includes, as we have seen in Sec. II, the interval $1.5<z<2$, corresponding to the birth of Lévy processes. However, we cannot rule out the possibility that the nonextensive formalism might turn out to be beneficial for another purpose, having to do with a form of out of equilibrium thermodynamics. According to Gaspard and Wang [10], the exponential sensitivity to the initial condition is recovered with extremely large values of the window size N. Is it possible to interpret N as a form of ''time''? Is it possible to assign a thermodynamic meaning to large portions of this extended region of transition to thermodynamics? The ''time'' meaning is enforced upon us by the fact that we are considering a dynamic condition corresponding to Lévy processes. In Sec. II we have seen that the Manneville map is a dynamic model generating Lévy processes. Thus we depart from the apparently safer perspective based on assigning to N the meaning of a volume, and we look at N as a form of time denoted by t. Furthermore, we have to point out that the condition $z>2$ implies a breakdown of the ordinary invariant distribution, or, as we shall see in Sec. IV, a relaxation to the invariant distribution with an infinite time scale. This is another reason to adopt an out of equilibrium perspective. Is the nonextensive formalism sensitive to this kind of phase transition?

For all these reasons, not only we look at the parameter N, for $N\gg 1$, as a continuous time t, but we explicitly set out of

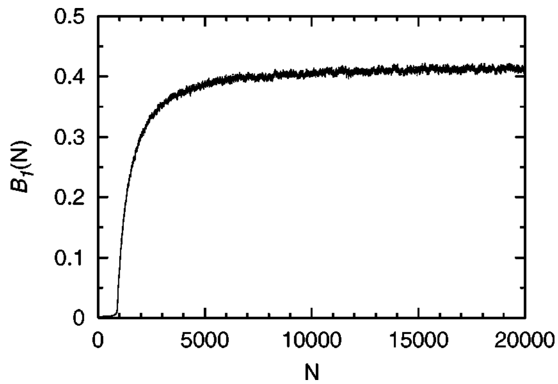

FIG. 3. $B_1(N)$ as a function of the iteration step N. $B_1(N)$ has a vanishing value until $N \simeq 900$, after which it undergoes an abrupt increase followed by a slow regression, with statistical fluctuations, to the constant value predicted by the Pesin theorem.

equilibrium conditions. Note that in the case where an invariant distribution exists, we can either make an average over the invariant distribution, as done with Eq. (7), or an average on time, as prescribed by Eqs. (5) and (6). We want to adopt a perspective that is valid in both cases, the case where the invariant distribution exists ($z \leqslant 2$) and the case where it does not ($z > 2$). This second condition will be discussed in detail in Sec. IV. To do this, we evaluate the nonextensive counterpart of the time dependent Lyapunov coefficent of Eq. (6), namely, the time dependent Lyapunov coefficent defined by Eq. (14), and we average it on an initial distribution, realized by the uniform distribution from $x=0$ to $x=\Delta<d$. This distribution is an out of equilibrium property, regardless of whether the invariant distribution exists or not. In other words, we apply our numerical treatment to an evaluation of

$$A_q(N) \equiv \langle \Lambda_q(N,x) \rangle_{ne}, \tag{27}$$

where $\langle \cdots \rangle_{ne}$ means the average on the out of equilbrium initial condition. All the numerical calculations of this section refer to $\Delta = 10^{-4}$. We select the value $z=1.7$ which is located in the region corresponding to the emergence of Lévy processes. We plan to study whether or not, in this explicitly out of equilibrium condition, a proper entropic index can be found that realizes a steady condition of entropy increase. This means a condition where the time derivative of $A_q(N)$ of Eq. (27) is constant. This means that we study

$$B_q(N) = A_q(N+1) - A_q(N). \tag{28}$$

The purpose of the numerical calculation is that of determining the time regions where $B_q(N)$ is constant. In Sec. II we have seen that in the case $z \leqslant 2$, the KS entropy exists and is finite. This means that a proper ''thermodynamic'' condition exists, this being the region of ordinary statistical mechanics with $Q=1$. However, we want to assess if there is room for nonextensive thermodynamics to apply as a form of nonequilibrium and transient thermodynamics. Futhermore, we want to assess the time duration of this form of non-extensive regime, corresponding to a given $Q \neq 1$, if it exists.

In Fig. 3 we plot the quantity $B_1(N)$. Note that $B_1(N)$ is the difference between two subsequent iteration steps of the Lyapunov coefficient $\Lambda(N,x)$ of Eq. (6) averaged over a nonequilibrium distribution. We see that a relatively extended time region shows up, ranging from $N=0$ to $N \simeq 900$, where $B_1(N)$ is approximately constant with an almost vanishing value. This is the region where according to the theory of Ref. [11], the nonextensive thermodynamics is expected to afford conceptual benefits, establishing a connection between the power law sensitivity to initial condition and the existence of a ''magic'' entropic index $Q>1$. In fact, the results of numerical calculations, for brevity not reported here, show, in accordance with Fig. 8 of Sec. IV, that $B_Q(N)$, with the ''magic'' Q of Eq. (25), is time independent. However, it is possible to show that this region is not characterized by any form of randomness, and consequently it is difficult to depict it as a thermodynamic region, even if reference is made to its hypothetical nonextensive nature.

To prove this important aspect, let us adopt the continoustime approximation to Eq. (15) and thus Eq. (17). As done, earlier let us denote by t the discrete value N when $N \gg 1$ applies, thereby implying that it can be considered as being continuous time. The solution of Eq. (17) for a trajectory with initial condition $x(0)$ is given by

$$x(t) = [x(0)^{1-z} - (z-1)t]^{1/(1-z)}. \tag{29}$$

Using this solution it is easy to find the time at which the first trajectory, that belonging to the right border of the initial distribution, exits from the laminar region. This time, denoted by T, is given by

$$T = \frac{d^{1-z}}{z-1}\left[\left(\frac{\Delta}{d}\right)^{1-z} - 1\right]. \tag{30}$$

Before this time no trajectory can exit from the laminar region and consequently no form of randomness can enter into play. Despite of the fact that the time interval $[0,T]$ is that corresponding to $B_Q(N)$, with Q given by Eq. (25), obtaining a constant nonvanishing value, it is hard to assign to this time regime a thermodynamic meaning, even if this is the Tsallis nonextensive thermodynamics.

It is interesting to evaluate analytically the time decrease of population $M(t)$ of the laminar region, namely, the time decrease of the number of trajectories that at time t are still found in the laminar region. We observe that for $t>T$ the initial condition $x(0)$ of the trajectory exiting at that time is given by

$$x(0) = [d^{1-z} - (z-1)t]^{1/(1-z)}. \tag{31}$$

This is easily proved by setting $x(t)$ of Eq. (29) equal to d. The population change dM in the infinitesimal time dt is given by

$$dM = M(0)\frac{dx(0)}{\Delta}, \tag{32}$$

where $M(0)$ is the number of trajectories within the laminar region at the initial time. Thus, using Eq. (32), we obtain

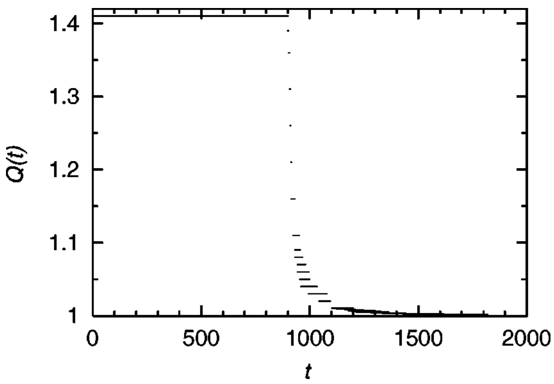

FIG. 4. Q as a function of time: The fast drop occurs at $t=T$. We see that at $t=T$, when the first trajectory escapes from the laminar region, $Q(t)$ drops from the value 1.41, established by Eq. (25), to the value 1.01, in about 200 iteration steps. This fast drop is followed by a much slower relaxation to the final value of 1, detailed in Fig. 5.

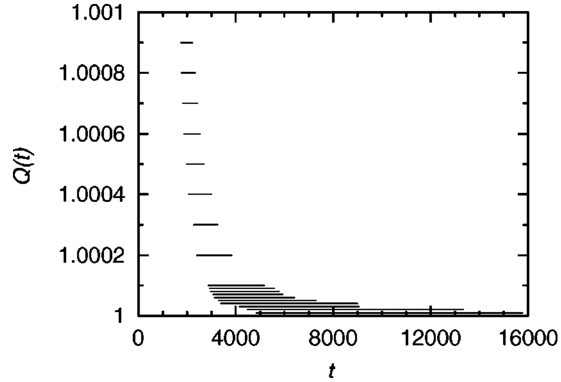

FIG. 5. Q as a function of time: Details of the slow regression following the fast drop. This figure shows, more clearly than Fig. 4, the windows of the entropy linear increase. The size of these windows becomes larger and larger at greater and greater times. The largest window shown here corresponds to $Q=1.00001$. In the scale of this figure, smaller values of q become indistinguishable from the value $q=1$, which corresponds to a window of infinite time size.

$$\frac{dM}{dt}=-\frac{d^z}{\Delta}\frac{M(0)}{[1+d^{z-1}(z-1)t]^{z/(z-1)}},\tag{33}$$

which makes it possible for us to establish the time evolution of the population at time t for $t>T$. In conclusion, we obtain

$$M(t)=M(0)\quad (t<T)\tag{34}$$

and

$$M(t)=M(0)\frac{d}{\Delta}\frac{1}{[1+d^{z-1}(z-1)t]^{1/z-1}}\quad (t>T).\tag{35}$$

It is interesting to remark that the time derivative of $M(t)$, at $t>T$, as resulting from Eq. (35), turns out to be proportional to the waiting time distribution $\psi(t)$ of Eq. (22). This means that $M(t)$, although depending on an arbitrary initial condition at times larger than T, reflects the stationary and statistical nature of $\psi(t)$. This fits the numerical observation made herein with the help of Fig. 6, that the process of regression to equilibrium is not affected by the return of the trajectories from the chaotic to the laminar region. We shall see, in fact, that the long-time behavior of $M(t)$ of Eq. (35) is a fair indicator of the process of regression to equilibrium. The numerical calculation agrees very well with the theoretical prediction of Eqs. (34) and (35). After a waiting time $T\approx 900$, the first exit occurs, $M(t)$ undergoes an abrupt decay followed by a slower regression to the final equilibrium condition. This is the process of memory erasure yielding, after a finite time scale, to Lévy diffusion. The time evolution of $M(t)$ is very similar to that of the time derivative of the Lyapunov coefficent of Eq. (13) with $q=Q=1.41$. For brevity, we omit showing this time evolution, that virtually coincides with the analytical results of Sec. IV (illustrated by Fig. 8).

In Figs. 4 and 5 we show the change of the entropic index from the prediction of Eq. (23), $Q=1+(z-1)/z$, corresponding to a trajectory motion not yet affected by the sporadic randomness, to the asymptotic value $Q=1$, produced by the repeated action of the chaotic region of the phase space. The calculation is done by searching, at any time $t>0$, for the value of q, realizing, temporarily, the condition of linear increase of the nonextensive entropy. This, in turn, is realized by looking for the value of q making $B_q(t)$ of Eq. (28) constant over windows of finite size. The window sizes are reported in both figures as small intervals, whose length tends to increase as the value of Q decreases. Figure 4 shows the remarkable fact that the transition from the region where only order exists to that where sporadic randomness begins to show up is signaled by a fast drop of the entropic index Q. Nevertheless, after the fast drop at $t=T$, the equilibrium value $Q=1$ is reached asymptotically in time with a slow regression process. This is illustrated by Fig. 5, which shows, more clearly than Fig. 4, that the time duration of this temporary nonextensive thermodynamics becomes larger and larger as Q comes closer and closer to the equilibrium value $Q=1$.

The process of regression of $Q(t)$ to the equilibrium value $Q=1$ is closely related to the process of transition from the initial unstable distribution to the final invariant measure. This latter process, in turn, is determined by the trajectories exiting the laminar region, and consequently is related to the time evolution $M(t)$ of Eqs. (34) and (35). However, an essential part of this process of relaxation to equilibrium might also be played by the trajectories that from the chaotic region are injected back into the laminar region. To establish whether this is true or not, it is convenient to monitor numerically the process of transition to equilibrium. This is done as follows. The interval [0,1] is divided into C cells of equal size. In our calculations we set $C=100$. We consider M trajectories with initial conditions uniformly distributed over the whole interval [0,1], and we iterate all of them N times. We set $M=10\,000$ and $N=100\,000$. In accordance with the prescription of Sec. III, we denote the time with the symbol t. At this stage we evaluate how many trajectories are found in a given cell with the label i. We call this number

M_i, and we set $P_i^{eq}=M_i/M$. This stage does not yet afford a proper numerical determination of the equilibrium distribution. This is so because with a finite number of trajectories M and cells of a finite size $1/C$, the quantity P_i^{eq} will turn out to be a fluctuating function of the iteration time $t\equiv N$. The intensity of these fluctuations depends on the selected values for the numbers C and M. To bypass this limitation we make a time average on τ further iterations after the time t, thereby defining

$$\langle P_i^{eq}\rangle_\tau \equiv \frac{1}{\tau}\sum_{t'=0}^{\tau} P_i(t'), \tag{36}$$

where $P_i(t')$ denotes the probability that a trajectory is found in the ith cell at the t'th iteration after time t. The result of this calculation is scarcely dependent on the value of τ, adopted if this is much greater than the time scale of the fluctuations of P_i^{eq}: In our case for $\tau > 500$ the right hand side of Eq. (36) remains practically constant for all the cells. Thus, we can omit the dependence of $\langle P_i^{eq}\rangle_\tau$ on τ and use Eq. (36), with a given value of $\tau > 500$, to define our numerical equilibrium distribution, which, for the sake of simplicity is again denoted by the symbol P_i^{eq}.

We are now in a position to properly address the important issue of regression to equilibrium. First of all, we adopt the same initial condition as that used in the earlier calculations. From this initial distribution we select a sample of M trajectories. At any time step we count how many trajectories are found in a given cell, thereby determining $P_i(t)$, that is, in this case, the probability that a trajectory is found in the cell with label i at time t. We compare this probability with the equilibrium probability, evaluated according to the earlier numerical prescription, thereby defining the variable $Y_i(t)$ as

$$Y_i(t) \equiv |P_i(t) - P_i^{eq}|. \tag{37}$$

Then we evaluate the relative dispersion $R_i(t)$ of the quantities $Y_i(t)$ around the equilibrium value P_i^{eq}:

$$R_i(t) = \frac{Y_i(t)}{P_i^{eq}}. \tag{38}$$

We now have to deal with the issue of fluctuations of $R_i(t)$ caused by the adoption of finite values for C and M. We follow the same procedure as that adopted to determine the equilibrium distribution. This means that we make the time average

$$\langle Y_i(t)\rangle_\tau = \frac{1}{\tau}\sum_{t'=t}^{t+\tau} Y_i(t'), \tag{39}$$

thereby deriving the time average of the relative dispersion $R_i(t)$:

$$\langle R_i(t)\rangle_\tau = \frac{\langle Y_i(t)\rangle_\tau}{P_i^{eq}}. \tag{40}$$

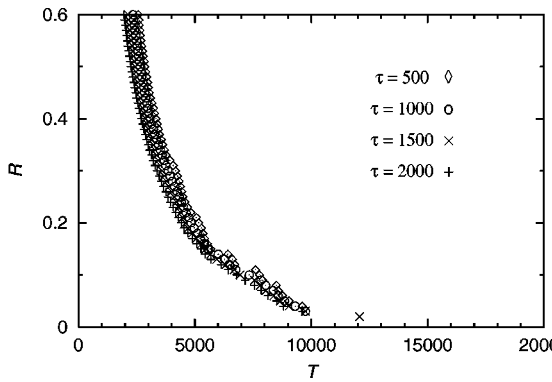

FIG. 6. R as a function of the time T, identified by T_{iter}. The number of cells and the trajectories used are $C=100$ and $M=10\,000$, respectively.

We then consider a set of numbers R of the interval [0,1]. For each of these numbers we determine the number of iterations T_{iter} necessary to make the time average of the relative dispersion [Eq. (40)], smaller than R, for all the C cells. This makes it possible for us to use the function $R(T_{iter})$ as a fair indicator of the relaxation to equilibrium. In fact, for any time T_{iter} we can say that the corresponding distribution departs from equilibrium by an amount of the order of $R \times 100\%$. The values of τ adopted range from $\tau=500$ to 3000. Within this wide interval the change of $R(T_{iter})$ is not significant.

In Fig. 6 we plot $R(T_{iter})$ for different values of τ. This figure shows that at $T_{iter}=10\,000$ and $T_{iter}=12\,000$ the distribution stills departs from equilibrium, for all the values of τ, by a quantity of the order of 3–4%. A smaller departure from equilibrium is recorded at times larger than 20 000, and therefore is not reported in Fig. 6. Consequently, according to the criterion we have adopted, at $T_{iter}=16\,000$ the departure from equilibrium is expected to be still of the order of 3–4%.

These numerical results prove that $M(t)$ is a good indicator of the process of relaxation to equilibrium. The trajectories injected back into the laminar region do not have any significant effect on determining the time scale of regression to equilibrium. Thus $Q(t)$ does not provide more information on the relaxation to equilibrium than $M(t)$. For example, the extended linearity window between $t \approx 6000$ and 16 000 with $Q=1.00001$, according to the prediction on $M(t)$ of Eq. (35), corresponds to a departure from equilibrium of the order of 3–4%. This is the same figure as that provided by Fig. 6. This has the effect of reducing the significance of $Q(t)$ as indicator of the regression to equilibrium. The function $M(t)$ has an analytical expression that affords indications as accurate as $Q(t)$. Furthermore, this casts doubts on the thermodynamic significance of the Tsallis entropic indicator in this context. In Sec. II, we have seen that the Tsallis nonextensive thermodynamics cannot be regarded as a form of equilibrium thermodynamics. The numerical results of this section lead us to conclude that even interpreting the nonextensive thermodynamics as a form of out of equilibrium thermodynamics is incorrect. In fact, the numerical results of this section prove that the relaxation to equilibrium does not depend on the trajectories injected back in the laminar region. The process of regression to equilibrium is essentially dictated by the deterministic dynamics driving the time evolution of $M(t)$. In this condition, invoking a recourse to thermodynamics, even of nonextensive nature, is not plausible on physical ground, since some degree of randomness seems to be essential to establish a thermodynamic perspective.

We have studied the time evolution of $Q(t)$ also in the case $z>2$, and we have not detected any significant change compared to the behavior discussed here, in spite of the fact that a sort of phase transition should occur when moving from $z<2$ to $z>2$. Rather than reporting the details for these further results, we shall try to shed light on the real source of entropy production with the analytical results of Sec. IV.

IV. ANALYTICAL RESULTS

This section is devoted to a discussion of analytical results based on a study of the continuous approximation to the Manneville map. The dynamic system under study will make it possible for us to establish the existence of a form of entropy which is expected to be very close to the KS entropy. We shall see that this is not exactly equivalent to the entropy. For the purpose of clarity, we define

$$\alpha(t)\equiv 1+(1-z)t \tag{41}$$

and

$$\beta\equiv\frac{1}{z-1},\quad \gamma\equiv\frac{z}{z-1}. \tag{42}$$

A. A solvable equation of motion mimicking intermittent behavior

The time evolution of the distribution density reads

$$\frac{\partial}{\partial t}\rho(x,t)=-\frac{\partial}{\partial x}(x^z\rho(x,t))+C(t). \tag{43}$$

Note that the function $C(t)$ is determined in such a way as to fulfill the condition that the the norm is conserved. This yields

$$C(t)\equiv\rho(1,t). \tag{44}$$

This equation can be thought of as a dynamical system on its own, consisting of the joint action of two processes, the former being deterministic and the latter random. The deterministic process is the solution of Eq. (17), which in turn corresponds to the first term on the right hand side of Eq. (43). The latter process corresponds to $C(t)$. This is the amount of randomness per unit of time. In fact, $C(t)$ is proportional to the number of trajectories injected back, per unit of time, into an initial condition x of the interval $[0.1]$. There is no dependence of $C(t)$ of x. This means that the process is totally random, and that the probability of obtaining x is uniform. Using the method of characteristics (for details on the method we refer the interested reader to Ref. [17], which adopted the same method to solve a problem quite similar to that here under study), we obtain:

$$\rho(x,t)=\int_0^t\frac{C(\tau)}{[(\alpha(t-\tau)-1)x^{z-1}]^{\gamma}}d\tau +\rho\left(\frac{x}{[(\alpha(t)-1)x^{z-1}]^{\beta}},0\right)\times\frac{1}{[(\alpha(t)-1)x^{z-1}]^{\gamma}}, \tag{45}$$

and, due to the definition of Eq. (44),

$$C(t)=\int_0^t\frac{C(\tau)}{(\alpha(t-\tau))^{\gamma}}d\tau+\rho\left(\frac{1}{(\alpha(t))^{\beta}},0\right)\frac{1}{(\alpha(t))^{\gamma}}. \tag{46}$$

Let us adopt the initial conditions used for the numerical calculation of Sec. III. The time T of Eq. (30) has to be written under the form

$$T=\beta[(\Delta)^{1-z}-1]. \tag{47}$$

It is straigthforward to show that $C(t)$ vanishes for $t<T$, and that at $t=T$ it makes a jump to the value

$$C(T)=\Delta^{z-1}. \tag{48}$$

The study of the more interesting condition $t>T$ can be done using the Laplace transform of Eq. (46), that yields

$$\hat{C}(s)=\frac{\hat{A}(s)}{1-\hat{f}(s)}, \tag{49}$$

where $\hat{C}(s)$ and $\hat{f}(s)$ denote the Laplace transforms of $C(t)$ and $f(t)\equiv 1/(1+(z-1)t)^{(z/(z-1))}$, respectively. The symbol $\hat{A}(t)$ denotes the Laplace transform of

$$A(t)\equiv\rho\left(\frac{1}{(\alpha(t))^{\beta}},0\right)\times\frac{1}{(\alpha(t))^{\gamma}}. \tag{50}$$

Thus

$$\hat{f}(s)\equiv\int_0^{+\infty}\frac{\exp(-st)}{(\alpha(t))^{\gamma}}dt \tag{51}$$

and

$$\hat{A}(s)=\frac{1}{\Delta}\int_T^{+\infty}\frac{\exp(-st)}{(\alpha(t))^{\gamma}}dt. \tag{52}$$

It is straightforward to prove that $\lim_{s\to 0}\hat{A}(s)=1$. We shall use this property in Sec. IV C.

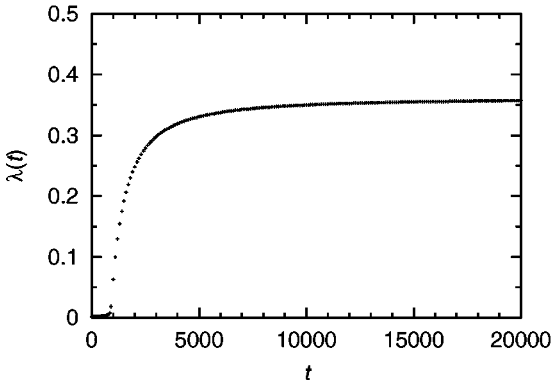

FIG. 7. $\lambda(t)$ as a function of t. The numerical calculation of this quantity is done using Eqs. (57) and (45).

B. Case $z<2$

Expression (51) yields (for small s)

$$\hat{f}(s) \approx 1 - \frac{s}{2-z}. \tag{53}$$

It is evident that this approximation is broken for $2 \leqslant z$. For this reason, in Sec. IV C we shall look for a different kind of expansion. Also using Eqs. (49) and (52), we arrive at

$$C(t) = (2-z)\left(1 - \frac{1}{\Delta}\frac{1}{(\alpha(t))^{\beta}}\right). \tag{54}$$

This is an interesting result. It means that

$$\lim_{t\to+\infty} C(t) = (2-z). \tag{55}$$

We have earlier seen that $C(t)$ monitors the occurrence of randomness per unit of time, at time t . The function $C(t)$, as we shall see, is a property closely related to the Kolmogorov complexity. In the case $z<2$, where we know that the KS entropy exists and is finite, this quantity also exists and is finite. It vanishes at $z=2$ precisely as the KS entropy does. Using Eq. (45) it is possible to derive an analytical expression for $\rho(t)$ which yields the invariant distribution through

$$\rho(x) \equiv \lim_{t\to+\infty} \rho(x,t) = \frac{2-z}{x^{z-1}}. \tag{56}$$

All this makes it possible to evaluate analytically the following time-dependent Lyapunov coefficient:

$$\lambda(t) \equiv \int_0^t \ln(1+zx^{z-1})\rho(x,t)dx. \tag{57}$$

Using Koopman's theorem [20] it is straigthforward to prove that this time dependent Lyapunov coefficient coincides with the quantity $B_1(N)$ (with $N \gg 1$, and set equal to t), discussed in Sec. III and illustrated in Fig. 3. Here we use Eqs. (57) and (45) to provide a further evaluation of $B_1(N)$. We call this quantity $\lambda(t)$, and we illustrate it in Fig. 7.

From Fig. 7 we see that the main properties of Fig. 3 are maintained. The only significant difference is the asymptotic limit, that here is evaluated using the invariant distribution of Eq. (56) rather than the numerical invariant distribution. It can be numerically assessed that the invariant distribution of the Manneville map coincides with Eq. (56) for $x \leqslant 0.2$. In the region $x>0.2$ the invariant distribution of the Manneville map obtains values larger than those provided by Eq. (56). The theory of this section allows us to establish a specific analytical form of the process of relaxation to equilibrium. Let us use the expression of Eq. (57) with a simplified expression for $\rho(x,t)$ of Eq. (45). The simplification rests on assuming that $C(t) \approx 2-z$. We then obtain

$$\lambda(t) \approx \int_0^1 \frac{2-z}{x^{z-1}}[1-((\alpha(t)-1)x^{z-1})^{-\beta}] \times \ln(1+zx^{z-1})dx. \tag{58}$$

Differentiating Eq. (58) with respect to time, and evaluating the resulting integral with the method of integratuion by parts, we obtain $d\lambda(t)/dt \approx t^{-z/(z-1)}$, which means

$$\lambda(t) \approx \alpha + t^{-\beta}. \tag{59}$$

Another interesting property that can be evaluated analytically is the Tsallis entropy of Eq. (2). Let us express this entropy in terms of the distribution entropy $\rho(x,t)$. It becomes

$$S_q(t) = \frac{1 - \int_0^t \rho(x,t)^q dx}{q-1}. \tag{60}$$

Using Eq. (45), we make this nonextensive entropy read

$$S_q(t) = \frac{1}{q-1}\left[1 - \int_0^{(\alpha(t))^{-\beta}} \Psi(\eta)(\zeta(t,\eta))^q d\eta\right], \tag{61}$$

where

$$\Psi(\eta) \equiv (1-(z-1)t\eta^{z-1})^{(q-1)\gamma}, \tag{62}$$

and

$$\zeta(t,\eta) \equiv \int_T^t \frac{C(\nu)}{(1-(z-1)\nu\eta^{z-1})^{\gamma}} d\nu + \rho(\eta,0). \tag{63}$$

It is straightforward to show that, for $t<T$,

$$S_q(t) = \frac{1}{q-1}\left[1 - \frac{1}{\Delta^q}\int_0^{\Delta} \Psi(\eta)d\eta\right]. \tag{64}$$

This expression confirms that the prescription of Eq. (25) is confined to a time scale smaller than T. The result illustrated by Fig. 8 is also of some interest. This is the time derivative of the entropy of Eqs. (61) and (63), corresponding to the magic value of Eq. (25) with $z=1.7$. We see that, in accordance with the results of Sec. III, it is constant in the time interval $[0,T]$, then it drops to a vanishing value with the

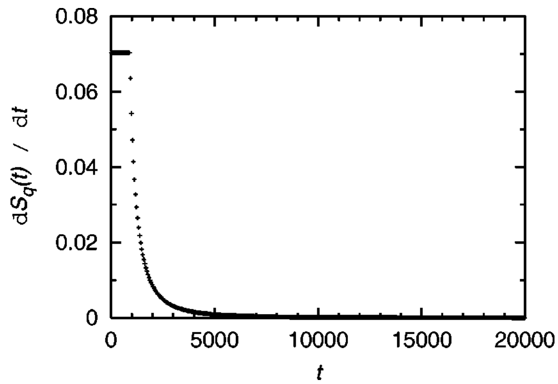

FIG. 8. Entropy increase per unit of time. This curve is obtained from the time derivative of $S_q(t)$ of Eq. (60). $z=1.7$.

inverse power law of Eq. (59). Note that the theoretical prediction is not the exact solution of Eq. (43). However, the accuracy of the resulting time evolution of $\rho(x,t)$ has been compared to the numerical solution of Eq. (43) and it has been found that the error is of the order of 5% and, of course, tends to vanish for $t\rightarrow\infty$. In conclusion, the analytical results of this section confirm the numerical results of Sec. III. The nonextensive nature of the Lévy processes advocated by the authors of Refs. [14–16] seem to conflict with the results of this paper. The process of regression of the Lyapunov coefficient to the constant value established by the Pesin theorem takes place on a finite time scale. Note, in fact, that the relaxation function of Eq. (59) is integrable for $z<2$.

At this stage only one problem is left. This has to do with whether or not the nonextensive thermodynamic approach might afford finite Lyapunov coefficients in the region $z>2$. As mentioned in Sec. III, the numerical analysis did not reveal any significant transition moving from $z<2$ to $z>2$. It is now the proper time of exploring this issue with analytical arguments.

C. Case $z>2$

To bypass the limitation of the expansion of Eq. (53), for $\hat{f}(s)$ of Eq. (51) we adopt the following expression:

$$\hat{f}(s)=(z-1)^{-\gamma}e^{\beta s}s^{\beta}\int_{\beta s}^{+\infty}\frac{\exp(-\eta)}{\eta^{\gamma}}d\eta. \tag{65}$$

After some algebra we make $\hat{f}(s)$ read

$$\hat{f}(s)=1-(z-1)^{-\beta}e^{\beta s}s^{\beta}[\Sigma(s)+\Gamma((z-2)\beta)], \tag{66}$$

where the function Σ is defined by the following expansion:

$$\Sigma(s)\equiv(z-1)^{-(z-2)\beta}\left[-\frac{1}{\beta(z-2)}s^{(z-2)\beta}+\frac{1}{2z-3}s^{(2z-3)\beta}+\cdots\right]. \tag{67}$$

Using Eqs. (49), (66), and (67), and the property noted in Sec. IV that $\lim_{s\rightarrow 0}\hat{A}(s)=1$, we obtain the important result

$$C(t)\approx\frac{\sin(\beta\pi)}{\pi}(z-1)^{\beta}t^{-(z-2)\beta}. \tag{68}$$

D. External entropy vs KS entropy

On the basis of the physical interpretation of the function $C(t)$ as the rate of the sporadic randomness per unit of time, we are led to define the complexity of the Manneville map as

$$K(t)\equiv\int_{T}^{t}C(t')dt'. \tag{69}$$

The results found in Secs. IV B and IV C lead us to conclude that in the asymptotic time limit $K(t)$ increases linearly with t for $z<2$ and as $t^{1/(z-1)}$ for $z>2$. It is remarkable that this coincides with the asymptotic behavior found by Gaspard and Wang [10] by means of their compression algorithm. This coincides also with the results of a more general compression algorithm developed by Argenti *et al.* [18]. All this is encouraging. However, the relation between $K(t)$ and the KS entropy is not quite clear. It is convenient to stress that this result can also be interpreted in the following way. Let us imagine that at regular intervals of time we draw a random number of the interval [0,1]. Let us call H the uncertainty associated with a single drawing, and let us define it as *internal entropy* per unit of time. We can conclude therefore that the internal entropy $S_I(N)$ is given by

$$S_I(N)=NH. \tag{70}$$

However, the observation of the random process refers to the *external* time defined by

$$t(N)=\tau_1+\cdots\tau_N. \tag{71}$$

We define *external* (E) entropy the internal entropy of Eq. (70) expressed in terms of the external rather than of the internal time. From the results of Sec. IV B we obtain that, for $z<2$, the E entropy S_E reads

$$S_E(t)=(2-z)Ht. \tag{72}$$

Is there a connection between the E entropy and the KS entropy? We note that the KS entropy has a totally dynamical definition, whereas the E entropy rests on the uncertainy H which is not clearly defined. However, if we consider the drawing of a number of the interval [0,1] equivalent to the KS entropy of the Bernouilli map, and we set $H=\ln 2$, and we compare the resulting expression for $S_E(t)$, divided by t, to h_{KS} of Eq. (19), we obtain that

$$R_E/h_{KS}=(\mu-1)/\mu, \tag{73}$$

where R_E is the rate of internal entropy per unit of time and $\mu=(z-1)/z$.

V. CONCLUDING REMARKS

The main result of this paper is that the extension of the connection between dynamics and thermodynamics proposed by the advocates of nonextensive thermodynamics [1–3] does not work in the case of intermittent processes. This approach does not afford any benefit in the region $z>2$, and does not even signal the transition from the stationary ($z<2$) to the nonstationary ($z>2$) regime. At the same time, there is no room left for the interpretation of Lévy processes as a form of nonextensive statistical mechanics, since the numerical results of Sec. III prove that the time regime where the Lévy processes show up is characterized by $Q=1$, which is a colorful way of saying that ordinary statistical mechanics apply. The analytical theory of Sec. IV proves that the main source of randomness is the crossing of the chaotic region. This explains why no significant benefit is derived from the adoption of a generalized form of Lyapunov coefficient.

These are negative results. The paper also contains a result positive. This is the analytical solution of the regression to equilibrium of the Manneville map. We note that this makes it possible for us to obtain, for the complexity properties of the Manneville map, the same conclusions as those of the earlier work of Gaspard and Wang [10], with no use of the mathematics of Kolmogorov and Gnedenko [19]. We think that the definition of the complexity of the Manneville map through the time integral of $C(t)$ affords a perspective whose exact connection with the KS entropy is worthy of further studies.

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